# Problem 4.42

**(a)** Using Equation 4.190 and the generalized Ehrenfest theorem (3.73), show that

$$\frac{d\langle \mathbf{r}\rangle}{dt} = \frac{1}{m}\langle(\mathbf{p} - q\mathbf{A})\rangle. \tag{4.192}$$

*Hint:* This stands for three equations—one for each component. Work it out for, say, the $x$ component, and then generalize your result.

**(b)** As always (see Equation 1.32) we identify $d\langle \mathbf{r}\rangle/dt$ with $\langle \mathbf{v}\rangle$. Show that[59]

$$m\frac{d\langle \mathbf{v}\rangle}{dt} = q\langle \mathbf{E}\rangle + \frac{q}{2m}\langle(\mathbf{p}\times\mathbf{B} - \mathbf{B}\times\mathbf{p})\rangle - \frac{q^2}{m}\langle(\mathbf{A}\times\mathbf{B})\rangle. \tag{4.193}$$

**(c)** In particular, if the fields $\mathbf{E}$ and $\mathbf{B}$ are *uniform* over the volume of the wave packet, show that

$$m\frac{d\langle \mathbf{v}\rangle}{dt} = q(\mathbf{E} + \langle \mathbf{v}\rangle \times \mathbf{B}), \tag{4.194}$$

so the *expectation value* of $\mathbf{v}$ moves according to the Lorentz force law, as we would expect from Ehrenfest's theorem.

[TYPOS: A factor of 2 is needed in front of $\mathbf{B}\times\mathbf{p}$ in Equation (4.193). Also, the footnote has a comma splice. Replace the comma before "but" with a semicolon or a period. This wave packet hasn't been introduced yet, so replace "the" with "a."]

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**Solution**

**Part (a)**

Calculate the time derivative of $\mathbf{r}$, the position vector.

$$\begin{aligned}
\frac{d\langle \mathbf{r}\rangle}{dt} &= \frac{d}{dt}\langle \Psi \,|\, \mathbf{r} \,|\, \Psi\rangle \\
&= \frac{d}{dt}\iiint\limits_{\text{all space}} \Psi^*\mathbf{r}\Psi\, d\mathcal{V} \\
&= \frac{d}{dt}\iiint\limits_{\text{all space}} \Psi^*\left(\sum_{j=1}^{3}\boldsymbol{\delta}_j x_j\right)\Psi\, d\mathcal{V} \\
&= \sum_{j=1}^{3}\boldsymbol{\delta}_j \iiint\limits_{\text{all space}} \frac{\partial}{\partial t}(\Psi^* x_j \Psi)\, d\mathcal{V} \\
&= \sum_{j=1}^{3}\boldsymbol{\delta}_j \iiint\limits_{\text{all space}} \left(\frac{\partial \Psi^*}{\partial t}x_j\Psi + \underbrace{\Psi^*\frac{\partial x_j}{\partial t}\Psi}_{=0} + \Psi^* x_j \frac{\partial \Psi}{\partial t}\right) d\mathcal{V}
\end{aligned}$$

[59]Note that $\mathbf{p}$ does not commute with $\mathbf{B}$, so $(\mathbf{p}\times\mathbf{B}) \neq -(\mathbf{B}\times\mathbf{p})$, but $\mathbf{A}$ *does* commute with $\mathbf{B}$, so $(\mathbf{A}\times\mathbf{B}) = -(\mathbf{B}\times\mathbf{A})$.

Use the Schrödinger equation to eliminate the time derivatives, and use the Hamiltonian for a particle with mass $m$, charge $q$, and momentum $\mathbf{p}$ in the presence of electromagnetic fields. Note that the Hamiltonian is hermitian (always), so $\hat{H}^\dagger = \hat{H}$. Recall that $\nabla(fg) = g\nabla f + f\nabla g$.

$$\begin{aligned}
\frac{d\langle\mathbf{r}\rangle}{dt} &= \sum_{j=1}^{3} \boldsymbol{\delta}_j \iiint\limits_{\text{all space}} \left[\left(\frac{1}{i\hbar}\hat{H}\Psi\right)^* x_j\Psi + \Psi^* x_j \left(\frac{1}{i\hbar}\hat{H}\Psi\right)\right] d\mathcal{V} \\
&= \sum_{j=1}^{3} \boldsymbol{\delta}_j \iiint\limits_{\text{all space}} \left[\left(\frac{1}{-i\hbar}\Psi^*\hat{H}^\dagger\right) x_j\Psi + \Psi^* x_j \left(\frac{1}{i\hbar}\hat{H}\Psi\right)\right] d\mathcal{V} \\
&= -\frac{1}{i\hbar}\sum_{j=1}^{3} \boldsymbol{\delta}_j \iiint\limits_{\text{all space}} \left[\Psi^*\hat{H}(x_j\Psi) - \Psi^* x_j\hat{H}\Psi\right] d\mathcal{V} \\
&= -\frac{1}{i\hbar}\sum_{j=1}^{3} \boldsymbol{\delta}_j \iiint\limits_{\text{all space}} \left\{\Psi^*\left[\frac{1}{2m}(\mathbf{p}-q\mathbf{A})^2 + q\varphi\right] x_j\Psi - \Psi^* x_j\left[\frac{1}{2m}(\mathbf{p}-q\mathbf{A})^2 + q\varphi\right]\Psi\right\} d\mathcal{V} \\
&= -\frac{1}{i\hbar}\sum_{j=1}^{3} \boldsymbol{\delta}_j \iiint\limits_{\text{all space}} \left\{\Psi^*\left[\frac{1}{2m}(-i\hbar\nabla-q\mathbf{A})^2 + q\varphi\right] x_j\Psi - \Psi^* x_j\left[\frac{1}{2m}(-i\hbar\nabla-q\mathbf{A})^2 + q\varphi\right]\Psi\right\} d\mathcal{V} \\
&= -\frac{1}{i\hbar}\sum_{j=1}^{3} \boldsymbol{\delta}_j \iiint\limits_{\text{all space}} \left\{\Psi^*\left[\frac{1}{2m}(i\hbar\nabla+q\mathbf{A})^2 + q\varphi\right] x_j\Psi - \Psi^* x_j\left[\frac{1}{2m}(i\hbar\nabla+q\mathbf{A})^2 + q\varphi\right]\Psi\right\} d\mathcal{V} \\
&= -\frac{1}{i\hbar}\sum_{j=1}^{3} \boldsymbol{\delta}_j \iiint\limits_{\text{all space}} \left[\frac{1}{2m}\Psi^*(i\hbar\nabla+q\mathbf{A})^2 x_j\Psi + \cancel{\Psi^* q\varphi x_j\Psi} - \frac{1}{2m}\Psi^* x_j(i\hbar\nabla+q\mathbf{A})^2\Psi - \cancel{\Psi^* x_j q\varphi\Psi}\right] d\mathcal{V} \\
&= -\frac{1}{2im\hbar}\sum_{j=1}^{3} \boldsymbol{\delta}_j \iiint\limits_{\text{all space}} \Big\{\Psi^*(i\hbar\nabla+q\mathbf{A})\cdot[i\hbar\nabla(x_j\Psi) + q\mathbf{A}(x_j\Psi)] \\
&\qquad - \Psi^* x_j(i\hbar\nabla+q\mathbf{A})\cdot(i\hbar\nabla\Psi + q\mathbf{A}\Psi)\Big\} d\mathcal{V} \\
&= -\frac{1}{2im\hbar}\sum_{j=1}^{3} \boldsymbol{\delta}_j \iiint\limits_{\text{all space}} \Big\{\Psi^*(i\hbar\nabla+q\mathbf{A})\cdot[i\hbar(\Psi\nabla x_j + x_j\nabla\Psi) + qx_j\Psi\mathbf{A}] \\
&\qquad - \Psi^* x_j[i^2\hbar^2(\nabla\cdot\nabla\Psi) + iq\hbar\nabla\cdot(\mathbf{A}\Psi) + iq\hbar(\mathbf{A}\cdot\nabla\Psi) + q^2(\mathbf{A}\cdot\mathbf{A})\Psi]\Big\} d\mathcal{V} \\
&= -\frac{1}{2im\hbar}\sum_{j=1}^{3} \boldsymbol{\delta}_j \iiint\limits_{\text{all space}} \Big\{\Psi^*(i\hbar\nabla+q\mathbf{A})\cdot(i\hbar\Psi\boldsymbol{\delta}_j + i\hbar x_j\nabla\Psi + qx_j\Psi\mathbf{A}) \\
&\qquad - \Psi^* x_j[-\hbar^2(\nabla^2\Psi) + iq\hbar\nabla\cdot(\mathbf{A}\Psi) + iq\hbar(\mathbf{A}\cdot\nabla\Psi) + q^2(\mathbf{A}\cdot\mathbf{A})\Psi]\Big\} d\mathcal{V}
\end{aligned}$$

Continue the simplification, noting that $\nabla \cdot (f\mathbf{C}) = f(\nabla \cdot \mathbf{C}) + \mathbf{C} \cdot \nabla f$.

$$\begin{aligned}
\frac{d\langle \mathbf{r}\rangle}{dt} &= -\frac{1}{2im\hbar}\sum_{j=1}^{3}\boldsymbol{\delta}_j \iiint\limits_{\text{all space}} \left\{ i\hbar\Psi^*\nabla\cdot(i\hbar\Psi\boldsymbol{\delta}_j + i\hbar x_j\nabla\Psi + qx_j\Psi\mathbf{A}) \right. \\
&\qquad + q\Psi^*\mathbf{A}\cdot(i\hbar\Psi\boldsymbol{\delta}_j + i\hbar x_j\nabla\Psi + qx_j\Psi\mathbf{A}) \\
&\qquad \left. - \Psi^* x_j[-\hbar^2(\nabla^2\Psi) + iq\hbar\nabla\cdot(\mathbf{A}\Psi) + iq\hbar(\mathbf{A}\cdot\nabla\Psi) + q^2(\mathbf{A}\cdot\mathbf{A})\Psi] \right\} d\mathcal{V} \\
&= -\frac{1}{2im\hbar}\sum_{j=1}^{3}\boldsymbol{\delta}_j \iiint\limits_{\text{all space}} \left[ i^2\hbar^2\Psi^*\nabla\cdot(\Psi\boldsymbol{\delta}_j) + i^2\hbar^2\Psi^*\nabla\cdot(x_j\nabla\Psi) + iq\hbar\Psi^*\nabla\cdot(x_j\Psi\mathbf{A}) \right. \\
&\qquad + iq\hbar\Psi^*(\mathbf{A}\cdot\boldsymbol{\delta}_j)\Psi + \cancel{iq\hbar\Psi^* x_j(\mathbf{A}\cdot\nabla\Psi)} + \cancel{q^2\Psi^* x_j\Psi(\mathbf{A}\cdot\mathbf{A})} \\
&\qquad \left. + \hbar^2\Psi^* x_j\nabla^2\Psi - iq\hbar\Psi^* x_j\nabla\cdot(\mathbf{A}\Psi) - \cancel{iq\hbar\Psi^* x_j(\mathbf{A}\cdot\nabla\Psi)} - \cancel{q^2\Psi^* x_j(\mathbf{A}\cdot\mathbf{A})\Psi} \right] d\mathcal{V} \\
&= -\frac{1}{2im\hbar}\sum_{j=1}^{3}\boldsymbol{\delta}_j \iiint\limits_{\text{all space}} \left\{ -\hbar^2\Psi^*[\Psi\overbrace{(\nabla\cdot\boldsymbol{\delta}_j)}^{=0} + \boldsymbol{\delta}_j\cdot\nabla\Psi] - \hbar^2\Psi^*[x_j(\nabla\cdot\nabla\Psi) + \nabla\Psi\cdot\nabla x_j] \right. \\
&\qquad + iq\hbar\Psi^*[x_j\Psi(\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot\nabla(x_j\Psi)] \\
&\qquad \left. + iq\hbar\Psi^* A_j\Psi + \hbar^2\Psi^* x_j\nabla^2\Psi - iq\hbar\Psi^* x_j[\Psi(\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot\nabla\Psi] \right\} d\mathcal{V} \\
&= -\frac{1}{2im\hbar}\sum_{j=1}^{3}\boldsymbol{\delta}_j \iiint\limits_{\text{all space}} \left\{ -\hbar^2\Psi^*(\nabla\Psi\cdot\boldsymbol{\delta}_j) - \cancel{\hbar^2\Psi^* x_j\nabla^2\Psi} - \hbar^2\Psi^*(\nabla\Psi\cdot\boldsymbol{\delta}_j) \right. \\
&\qquad + \cancel{iq\hbar\Psi^* x_j\Psi(\nabla\cdot\mathbf{A})} + iq\hbar\Psi^*[\mathbf{A}\cdot\nabla(x_j\Psi)] \\
&\qquad \left. + iq\hbar\Psi^* A_j\Psi + \cancel{\hbar^2\Psi^* x_j\nabla^2\Psi} - \cancel{iq\hbar\Psi^* x_j\Psi(\nabla\cdot\mathbf{A})} - iq\hbar\Psi^* x_j\mathbf{A}\cdot\nabla\Psi \right\} d\mathcal{V} \\
&= -\frac{1}{2im\hbar}\sum_{j=1}^{3}\boldsymbol{\delta}_j \iiint\limits_{\text{all space}} \left[ -2\hbar^2\Psi^*(\nabla\Psi\cdot\boldsymbol{\delta}_j) + iq\hbar\Psi^*\mathbf{A}\cdot(\Psi\nabla x_j + x_j\nabla\Psi) + iq\hbar\Psi^* A_j\Psi - iq\hbar\Psi^* x_j\mathbf{A}\cdot\nabla\Psi \right] d\mathcal{V} \\
&= -\frac{1}{2im\hbar}\sum_{j=1}^{3}\boldsymbol{\delta}_j \iiint\limits_{\text{all space}} \left[ -2\hbar^2\Psi^*\frac{\partial\Psi}{\partial x_j} + iq\hbar\Psi^*\mathbf{A}\cdot(\Psi\nabla x_j) + \cancel{iq\hbar\Psi^* x_j\mathbf{A}\cdot\nabla\Psi} \right. \\
&\qquad \left. + iq\hbar\Psi^* A_j\Psi - \cancel{iq\hbar\Psi^* x_j\mathbf{A}\cdot\nabla\Psi} \right] d\mathcal{V}
\end{aligned}$$

Continue the simplification.

$$\begin{aligned}
\frac{d\langle \mathbf{r}\rangle}{dt} &= -\frac{1}{2im\hbar}\sum_{j=1}^{3}\boldsymbol{\delta}_j \iiint\limits_{\text{all space}} \left[-2\hbar^2\Psi^*\frac{\partial\Psi}{\partial x_j} + iq\hbar\Psi^*\mathbf{A}\cdot(\Psi\boldsymbol{\delta}_j) + iq\hbar\Psi^*A_j\Psi\right] d\mathcal{V} \\
&= -\frac{1}{2im\hbar}\sum_{j=1}^{3}\boldsymbol{\delta}_j \iiint\limits_{\text{all space}} \left[-2\hbar^2\Psi^*\frac{\partial\Psi}{\partial x_j} + iq\hbar\Psi^*(\Psi A_j) + iq\hbar\Psi^*A_j\Psi\right] d\mathcal{V} \\
&= -\frac{1}{2im\hbar}\sum_{j=1}^{3}\boldsymbol{\delta}_j \iiint\limits_{\text{all space}} \left(-2\hbar^2\Psi^*\frac{\partial\Psi}{\partial x_j} + 2iq\hbar\Psi^*A_j\Psi\right) d\mathcal{V} \\
&= \frac{1}{m}\sum_{j=1}^{3}\boldsymbol{\delta}_j \iiint\limits_{\text{all space}} \left(\frac{\hbar}{i}\Psi^*\frac{\partial\Psi}{\partial x_j} - q\Psi^*A_j\Psi\right) d\mathcal{V} \\
&= \frac{1}{m}\iiint\limits_{\text{all space}} \left[\frac{\hbar}{i}\Psi^*\left(\sum_{j=1}^{3}\boldsymbol{\delta}_j\frac{\partial\Psi}{\partial x_j}\right) - q\Psi^*\left(\sum_{j=1}^{3}\boldsymbol{\delta}_jA_j\right)\Psi\right] d\mathcal{V} \\
&= \frac{1}{m}\iiint\limits_{\text{all space}} \left[\frac{\hbar}{i}\Psi^*(\nabla\Psi) - q\Psi^*\mathbf{A}\Psi\right] d\mathcal{V} \\
&= \frac{1}{m}\iiint\limits_{\text{all space}} [\Psi^*(-i\hbar\nabla)\Psi - \Psi^*(q\mathbf{A})\Psi]\, d\mathcal{V} \\
&= \frac{1}{m}\iiint\limits_{\text{all space}} \Psi^*[(-i\hbar\nabla) - (q\mathbf{A})]\Psi\, d\mathcal{V} \\
&= \frac{1}{m}\iiint\limits_{\text{all space}} \Psi^*(\mathbf{p} - q\mathbf{A})\Psi\, d\mathcal{V} \\
&= \frac{1}{m}\langle\Psi\,|\,\mathbf{p} - q\mathbf{A}\,|\,\Psi\rangle
\end{aligned}$$

Therefore,

$$\frac{d\langle \mathbf{r}\rangle}{dt} = \frac{1}{m}\langle \mathbf{p} - q\mathbf{A}\rangle.$$

**Part (b)**

Begin with the result of part (a), replace $d\langle\mathbf{r}\rangle/dt$ with $\langle\mathbf{v}\rangle$, multiply both sides by $m$, and differentiate both sides with respect to $t$.

$$\begin{aligned}
\frac{d\langle\mathbf{r}\rangle}{dt} &= \frac{1}{m}\langle\mathbf{p}-q\mathbf{A}\rangle \\
\langle\mathbf{v}\rangle &= \frac{1}{m}\langle\mathbf{p}-q\mathbf{A}\rangle \\
m\langle\mathbf{v}\rangle &= \langle\mathbf{p}-q\mathbf{A}\rangle \\
\frac{d}{dt}\left(m\langle\mathbf{v}\rangle\right) &= \frac{d}{dt}\langle\mathbf{p}-q\mathbf{A}\rangle \\
m\frac{d\langle\mathbf{v}\rangle}{dt} &= \frac{d}{dt}\langle\Psi\,|\,\mathbf{p}-q\mathbf{A}\,|\,\Psi\rangle \\
&= \frac{d}{dt}\iiint\limits_{\text{all space}}\Psi^*\left[\left(\frac{\hbar}{i}\nabla\right)-(q\mathbf{A})\right]\Psi\,d\mathcal{V} \\
&= \frac{d}{dt}\iiint\limits_{\text{all space}}\left[(-i\hbar)\Psi^*\nabla\Psi-q\Psi^*\mathbf{A}\Psi\right]d\mathcal{V} \\
&= \iiint\limits_{\text{all space}}\frac{\partial}{\partial t}\left[(-i\hbar)\Psi^*\nabla\Psi-q\Psi^*\mathbf{A}\Psi\right]d\mathcal{V} \\
&= -i\hbar\iiint\limits_{\text{all space}}\frac{\partial}{\partial t}(\Psi^*\nabla\Psi)\,d\mathcal{V} \\
&\quad -q\iiint\limits_{\text{all space}}\frac{\partial}{\partial t}(\Psi^*\mathbf{A}\Psi)\,d\mathcal{V} \\
&= -i\hbar\iiint\limits_{\text{all space}}\left[\frac{\partial\Psi^*}{\partial t}\nabla\Psi+\Psi^*\nabla\left(\frac{\partial\Psi}{\partial t}\right)\right]d\mathcal{V} \\
&\quad -q\iiint\limits_{\text{all space}}\left(\frac{\partial\Psi^*}{\partial t}\mathbf{A}\Psi+\Psi^*\frac{\partial\mathbf{A}}{\partial t}\Psi+\Psi^*\mathbf{A}\frac{\partial\Psi}{\partial t}\right)d\mathcal{V} \\
&= -i\hbar\iiint\limits_{\text{all space}}\left[\left(\frac{1}{i\hbar}\hat{H}\Psi\right)^*\nabla\Psi+\Psi^*\nabla\left(\frac{1}{i\hbar}\hat{H}\Psi\right)\right]d\mathcal{V} \\
&\quad -q\iiint\limits_{\text{all space}}\left[\left(\frac{1}{i\hbar}\hat{H}\Psi\right)^*\mathbf{A}\Psi+\Psi^*\frac{\partial\mathbf{A}}{\partial t}\Psi+\Psi^*\mathbf{A}\left(\frac{1}{i\hbar}\hat{H}\Psi\right)\right]d\mathcal{V}
\end{aligned}$$

Take the complex conjugate and substitute the appropriate Hamiltonian.

$$m\frac{d\langle\mathbf{v}\rangle}{dt} = -i\hbar \iiint_{\text{all space}} \left[\left(\frac{1}{-i\hbar}\Psi^*\hat{H}^\dagger\right)\nabla\Psi + \Psi^*\nabla\left(\frac{1}{i\hbar}\hat{H}\Psi\right)\right]d\mathcal{V}$$

$$- q\iiint_{\text{all space}} \left[\left(\frac{1}{-i\hbar}\Psi^*\hat{H}^\dagger\right)\mathbf{A}\Psi + \Psi^*\frac{\partial\mathbf{A}}{\partial t}\Psi + \Psi^*\mathbf{A}\left(\frac{1}{i\hbar}\hat{H}\Psi\right)\right]d\mathcal{V}$$

$$= \iiint_{\text{all space}} \left[\Psi^*\hat{H}(\nabla\Psi) - \Psi^*\nabla(\hat{H}\Psi)\right]d\mathcal{V}$$

$$+ \frac{q}{i\hbar}\iiint_{\text{all space}} \left[\Psi^*\hat{H}(\mathbf{A}\Psi) - \Psi^*\mathbf{A}(\hat{H}\Psi)\right]d\mathcal{V}$$

$$- q\iiint_{\text{all space}} \Psi^*\frac{\partial\mathbf{A}}{\partial t}\Psi\,d\mathcal{V}$$

$$= \iiint_{\text{all space}} \left\{\Psi^*\left[\frac{1}{2m}(\mathbf{p}-q\mathbf{A})^2 + q\varphi\right]\nabla\Psi - \Psi^*\nabla\left[\frac{1}{2m}(\mathbf{p}-q\mathbf{A})^2 + q\varphi\right]\Psi\right\}d\mathcal{V}$$

$$+ \frac{q}{i\hbar}\iiint_{\text{all space}} \left\{\Psi^*\left[\frac{1}{2m}(\mathbf{p}-q\mathbf{A})^2 + q\varphi\right]\mathbf{A}\Psi - \Psi^*\mathbf{A}\left[\frac{1}{2m}(\mathbf{p}-q\mathbf{A})^2 + q\varphi\right]\Psi\right\}d\mathcal{V}$$

$$- q\iiint_{\text{all space}} \Psi^*\frac{\partial\mathbf{A}}{\partial t}\Psi\,d\mathcal{V}$$

$$= \iiint_{\text{all space}} \left\{\Psi^*\left[\frac{1}{2m}(-i\hbar\nabla-q\mathbf{A})^2 + q\varphi\right]\nabla\Psi - \Psi^*\nabla\left[\frac{1}{2m}(-i\hbar\nabla-q\mathbf{A})^2 + q\varphi\right]\Psi\right\}d\mathcal{V}$$

$$+ \frac{q}{i\hbar}\iiint_{\text{all space}} \left\{\Psi^*\left[\frac{1}{2m}(-i\hbar\nabla-q\mathbf{A})^2 + q\varphi\right]\mathbf{A}\Psi - \Psi^*\mathbf{A}\left[\frac{1}{2m}(-i\hbar\nabla-q\mathbf{A})^2 + q\varphi\right]\Psi\right\}d\mathcal{V}$$

$$- q\iiint_{\text{all space}} \Psi^*\frac{\partial\mathbf{A}}{\partial t}\Psi\,d\mathcal{V}$$

$$= \iiint_{\text{all space}} \left\{\Psi^*\left[\frac{1}{2m}(i\hbar\nabla+q\mathbf{A})^2 + q\varphi\right]\nabla\Psi - \Psi^*\nabla\left[\frac{1}{2m}(i\hbar\nabla+q\mathbf{A})^2 + q\varphi\right]\Psi\right\}d\mathcal{V}$$

$$+ \frac{q}{i\hbar}\iiint_{\text{all space}} \left\{\Psi^*\left[\frac{1}{2m}(i\hbar\nabla+q\mathbf{A})^2 + q\varphi\right]\mathbf{A}\Psi - \Psi^*\mathbf{A}\left[\frac{1}{2m}(i\hbar\nabla+q\mathbf{A})^2 + q\varphi\right]\Psi\right\}d\mathcal{V}$$

$$- q\iiint_{\text{all space}} \Psi^*\frac{\partial\mathbf{A}}{\partial t}\Psi\,d\mathcal{V}$$

Continue the simplification, noting that $\nabla(\varphi\Psi) = \varphi\nabla\Psi + \Psi\nabla\varphi$.

$$\begin{aligned}
m\frac{d\langle\mathbf{v}\rangle}{dt} &= \iiint\limits_{\text{all space}} \left\{\frac{1}{2m}\Psi^*(i\hbar\nabla + q\mathbf{A})\cdot[i\hbar\nabla(\nabla\Psi) + q\mathbf{A}(\nabla\Psi)] + \Psi^* q\varphi\nabla\Psi - \frac{1}{2m}\Psi^*\nabla(i\hbar\nabla + q\mathbf{A})\cdot(i\hbar\nabla\Psi + q\mathbf{A}\Psi) - q\Psi^*\nabla(\varphi\Psi)\right\} d\mathcal{V} \\
&\quad + \frac{q}{i\hbar}\iiint\limits_{\text{all space}} \left\{\frac{1}{2m}\Psi^*(i\hbar\nabla + q\mathbf{A})\cdot[i\hbar\nabla(\mathbf{A}\Psi) + q\mathbf{A}(\mathbf{A}\Psi)] + \cancel{\Psi^* q\varphi(\mathbf{A}\Psi)} - \frac{1}{2m}\Psi^*\mathbf{A}(i\hbar\nabla + q\mathbf{A})\cdot(i\hbar\nabla\Psi + q\mathbf{A}\Psi) - \cancel{q\Psi^*\mathbf{A}\varphi\Psi}\right\} d\mathcal{V} \\
&\quad - q\iiint\limits_{\text{all space}} \Psi^*\frac{\partial\mathbf{A}}{\partial t}\Psi\, d\mathcal{V} \\
&= \iiint\limits_{\text{all space}} \left\{\frac{1}{2m}\Psi^*(i\hbar\nabla + q\mathbf{A})\cdot[i\hbar\nabla(\nabla\Psi) + q\mathbf{A}(\nabla\Psi)] + \cancel{\Psi^* q\varphi\nabla\Psi} - \frac{1}{2m}\Psi^*\nabla(i\hbar\nabla + q\mathbf{A})\cdot(i\hbar\nabla\Psi + q\mathbf{A}\Psi) - \cancel{q\Psi^*\varphi\nabla\Psi} - q\Psi^*(\nabla\varphi)\Psi\right\} d\mathcal{V} \\
&\quad + \frac{q}{2im\hbar}\iiint\limits_{\text{all space}} \left\{\Psi^*(i\hbar\nabla + q\mathbf{A})\cdot[i\hbar\nabla(\mathbf{A}\Psi) + q\mathbf{A}(\mathbf{A}\Psi)] - \Psi^*\mathbf{A}(i\hbar\nabla + q\mathbf{A})\cdot(i\hbar\nabla\Psi + q\mathbf{A}\Psi)\right\} d\mathcal{V} \\
&\quad - q\iiint\limits_{\text{all space}} \Psi^*\frac{\partial\mathbf{A}}{\partial t}\Psi\, d\mathcal{V} \\
&= \frac{1}{2m}\iiint\limits_{\text{all space}} \left\{\Psi^*(i\hbar\nabla + q\mathbf{A})\cdot[i\hbar\nabla(\nabla\Psi) + q\mathbf{A}(\nabla\Psi)] - \Psi^*\nabla(i\hbar\nabla + q\mathbf{A})\cdot(i\hbar\nabla\Psi + q\mathbf{A}\Psi)\right\} d\mathcal{V} \\
&\quad + \frac{q}{2im\hbar}\iiint\limits_{\text{all space}} \left\{\Psi^*(i\hbar\nabla + q\mathbf{A})\cdot[i\hbar\nabla(\mathbf{A}\Psi) + q\mathbf{A}(\mathbf{A}\Psi)] - \Psi^*\mathbf{A}[-\hbar^2(\nabla\cdot\nabla\Psi) + iq\hbar\nabla\cdot(\mathbf{A}\Psi) + iq\hbar(\mathbf{A}\cdot\nabla\Psi) + q^2(\mathbf{A}\cdot\mathbf{A})\Psi]\right\} d\mathcal{V} \\
&\quad + q\iiint\limits_{\text{all space}} \Psi^*\left(-\nabla\varphi - \frac{\partial\mathbf{A}}{\partial t}\right)\Psi\, d\mathcal{V}
\end{aligned}$$

Continue the simplification, noting that $\mathbf{E} = -\nabla\varphi - \partial\mathbf{A}/\partial t$.

$$m\frac{d\langle\mathbf{v}\rangle}{dt} = \frac{1}{2m}\iiint\limits_{\text{all space}} \left\{\Psi^*(i\hbar\nabla + q\mathbf{A})\cdot[i\hbar\nabla(\nabla\Psi) + q\mathbf{A}(\nabla\Psi)] - \Psi^*\nabla[-\hbar^2(\nabla\cdot\nabla\Psi) + iq\hbar\nabla\cdot(\mathbf{A}\Psi) + iq\hbar(\mathbf{A}\cdot\nabla\Psi) + q^2(\mathbf{A}\cdot\mathbf{A})\Psi]\right\} d\mathcal{V}$$

$$+ \frac{q}{2im\hbar}\iiint\limits_{\text{all space}} \left\{\Psi^*(i\hbar\nabla + q\mathbf{A})\cdot[i\hbar\nabla(\mathbf{A}\Psi) + q\mathbf{A}(\mathbf{A}\Psi)] - \Psi^*\mathbf{A}[-\hbar^2(\nabla\cdot\nabla\Psi) + iq\hbar\nabla\cdot(\mathbf{A}\Psi) + iq\hbar(\mathbf{A}\cdot\nabla\Psi) + q^2(\mathbf{A}\cdot\mathbf{A})\Psi]\right\} d\mathcal{V}$$

$$+ q\iiint\limits_{\text{all space}} \Psi^*\mathbf{E}\Psi\, d\mathcal{V}$$

$$= \frac{1}{2m}\iiint\limits_{\text{all space}} \left\{\Psi^*(i\hbar\nabla + q\mathbf{A})\cdot[i\hbar\nabla(\nabla\Psi) + q\mathbf{A}(\nabla\Psi)] + \hbar^2\Psi^*\nabla(\nabla^2\Psi) - iq\hbar\Psi^*\nabla[\nabla\cdot(\mathbf{A}\Psi)] - iq\hbar\Psi^*\nabla(\mathbf{A}\cdot\nabla\Psi) - q^2\Psi^*\nabla(A^2\Psi)\right\} d\mathcal{V}$$

$$+ \frac{q}{2im\hbar}\iiint\limits_{\text{all space}} \left\{\Psi^*(i\hbar\nabla + q\mathbf{A})\cdot[i\hbar\nabla(\mathbf{A}\Psi) + q\mathbf{A}(\mathbf{A}\Psi)] + \hbar^2\Psi^*\mathbf{A}\nabla^2\Psi - iq\hbar\Psi^*\mathbf{A}[\nabla\cdot(\mathbf{A}\Psi)] - iq\hbar\Psi^*\mathbf{A}(\mathbf{A}\cdot\nabla\Psi) - q^2A^2\Psi^*\mathbf{A}\Psi\right\} d\mathcal{V}$$

$$+ q\langle\Psi\,|\,\mathbf{E}\,|\,\Psi\rangle$$

$$= \frac{1}{2m}\iiint\limits_{\text{all space}} \left\{\Psi^*\{-\hbar^2[\nabla\cdot\nabla(\nabla\Psi)] + iq\hbar\nabla\cdot[\mathbf{A}(\nabla\Psi)] + iq\hbar\mathbf{A}\cdot[\nabla(\nabla\Psi)] + q^2\mathbf{A}\cdot(\mathbf{A}\nabla\Psi)\}\right.$$

$$\left. + \hbar^2\Psi^*\nabla(\nabla^2\Psi) - iq\hbar\Psi^*\nabla[\nabla\cdot(\mathbf{A}\Psi)] - iq\hbar\Psi^*\nabla(\mathbf{A}\cdot\nabla\Psi) - q^2\Psi^*\nabla(A^2\Psi)\right\} d\mathcal{V}$$

$$+ \frac{q}{2im\hbar}\iiint\limits_{\text{all space}} \left\{\Psi^*\{-\hbar^2[\nabla\cdot\nabla(\mathbf{A}\Psi)] + iq\hbar[\nabla\cdot(\mathbf{A}\mathbf{A}\Psi)] + iq\hbar[\mathbf{A}\cdot\nabla(\mathbf{A}\Psi)] + q^2[\mathbf{A}\cdot(\mathbf{A}\mathbf{A}\Psi)]\}\right.$$

$$\left. + \hbar^2\Psi^*\mathbf{A}\nabla^2\Psi - iq\hbar\Psi^*\mathbf{A}[\nabla\cdot(\mathbf{A}\Psi)] - iq\hbar\Psi^*\mathbf{A}(\mathbf{A}\cdot\nabla\Psi) - q^2A^2\Psi^*\mathbf{A}\Psi\right\} d\mathcal{V}$$

$$+ q\langle\mathbf{E}\rangle$$

Continue the simplification.

$$m\frac{d\langle \mathbf{v}\rangle}{dt} = \frac{1}{2m}\iiint\limits_{\text{all space}} \left\{ \cancel{-\hbar^2\Psi^*\nabla^2(\nabla\Psi)} + iq\hbar\Psi^*\nabla\cdot[\mathbf{A}(\nabla\Psi)] + iq\hbar\Psi^*\mathbf{A}\cdot[\nabla(\nabla\Psi)] + q^2\Psi^*\mathbf{A}\cdot(\mathbf{A}\nabla\Psi) \right.$$

$$\left. + \cancel{\hbar^2\Psi^*\nabla(\nabla^2\Psi)} - iq\hbar\Psi^*\nabla[\nabla\cdot(\mathbf{A}\Psi)] - iq\hbar\Psi^*\nabla(\mathbf{A}\cdot\nabla\Psi) - q^2\Psi^*\nabla(A^2\Psi) \right\} d\mathcal{V}$$

$$+ \frac{q}{2im\hbar}\iiint\limits_{\text{all space}} \left\{ -\hbar^2\Psi^*\nabla^2(\mathbf{A}\Psi) + iq\hbar\Psi^*[\nabla\cdot(\mathbf{A}\mathbf{A}\Psi)] + iq\hbar\Psi^*[\mathbf{A}\cdot\nabla(\mathbf{A}\Psi)] + \cancel{q^2\Psi^*[\mathbf{A}\cdot(\mathbf{A}\mathbf{A}\Psi)]} \right.$$

$$\left. + \hbar^2\Psi^*\mathbf{A}\nabla^2\Psi - iq\hbar\Psi^*\mathbf{A}[\nabla\cdot(\mathbf{A}\Psi)] - iq\hbar\Psi^*\mathbf{A}(\mathbf{A}\cdot\nabla\Psi) - \cancel{q^2A^2\Psi^*\mathbf{A}\Psi} \right\} d\mathcal{V}$$

$$+ q\langle \mathbf{E}\rangle$$

These terms cancel because

$$\nabla^2(\nabla\Psi) = \left(\sum_{j=1}^3 \frac{\partial^2}{\partial x_j^2}\right)\left(\sum_{k=1}^3 \boldsymbol{\delta}_k \frac{\partial\Psi}{\partial x_k}\right) = \sum_{j=1}^3\sum_{k=1}^3 \boldsymbol{\delta}_k \frac{\partial^2}{\partial x_j^2}\left(\frac{\partial\Psi}{\partial x_k}\right) = \sum_{j=1}^3\sum_{k=1}^3 \boldsymbol{\delta}_k \frac{\partial}{\partial x_k}\left(\frac{\partial^2\Psi}{\partial x_j^2}\right) = \sum_{k=1}^3 \boldsymbol{\delta}_k \frac{\partial}{\partial x_k}\left(\sum_{j=1}^3 \frac{\partial^2\Psi}{\partial x_j^2}\right) = \nabla(\nabla^2\Psi),$$

and

$$\mathbf{A}\cdot(\mathbf{A}\mathbf{A}\Psi) = \left(\sum_{j=1}^3 \boldsymbol{\delta}_j A_j\right)\cdot\left[\left(\sum_{k=1}^3 \boldsymbol{\delta}_k A_k\right)\left(\sum_{l=1}^3 \boldsymbol{\delta}_l A_l\right)\Psi\right] = \left(\sum_{j=1}^3 \boldsymbol{\delta}_j A_j\right)\cdot\left(\sum_{k=1}^3\sum_{l=1}^3 \boldsymbol{\delta}_k\boldsymbol{\delta}_l A_k A_l\Psi\right) = \sum_{j=1}^3\sum_{k=1}^3\sum_{l=1}^3 \overbrace{(\boldsymbol{\delta}_j\cdot\boldsymbol{\delta}_k)}^{=\,\delta_{jk}}\boldsymbol{\delta}_l A_j A_k A_l\Psi$$

$$= \sum_{k=1}^3\sum_{l=1}^3 \boldsymbol{\delta}_l A_k A_k A_l \Psi$$

$$= \left(\sum_{k=1}^3 A_k^2\right)\left(\sum_{l=1}^3 \boldsymbol{\delta}_l A_l\right)\Psi = A^2\mathbf{A}\Psi.$$

Continue the simplification, grouping the terms with $iq\hbar$ and $q^2$ separately.

$$\begin{aligned} m\frac{d\langle\mathbf{v}\rangle}{dt} &= \frac{1}{2m}\iiint_{\text{all space}}\left\{iq\hbar\Psi^*\nabla\cdot[\mathbf{A}(\nabla\Psi)] + iq\hbar\Psi^*\mathbf{A}\cdot[\nabla(\nabla\Psi)] + q^2\Psi^*\mathbf{A}\cdot(\mathbf{A}\nabla\Psi)\right. \\ &\qquad \left. - iq\hbar\Psi^*\nabla[\nabla\cdot(\mathbf{A}\Psi)] - iq\hbar\Psi^*\nabla(\mathbf{A}\cdot\nabla\Psi) - q^2\Psi^*\nabla(A^2\Psi)\right\}d\mathcal{V} \\ &\quad + \frac{1}{2m}\iiint_{\text{all space}}\left\{iq\hbar\Psi^*\nabla^2(\mathbf{A}\Psi) + q^2\Psi^*[\nabla\cdot(\mathbf{A}\mathbf{A}\Psi)] + q^2\Psi^*[\mathbf{A}\cdot\nabla(\mathbf{A}\Psi)]\right. \\ &\qquad \left. - iq\hbar\Psi^*\mathbf{A}\nabla^2\Psi - q^2\Psi^*\mathbf{A}[\nabla\cdot(\mathbf{A}\Psi)] - q^2\Psi^*\mathbf{A}(\mathbf{A}\cdot\nabla\Psi)\right\}d\mathcal{V} \\ &\quad + q\langle\mathbf{E}\rangle \\ &= \frac{iq\hbar}{2m}\iiint_{\text{all space}}\Psi^*\left\{\nabla\cdot[\mathbf{A}(\nabla\Psi)] + \mathbf{A}\cdot[\nabla(\nabla\Psi)] + \nabla^2(\mathbf{A}\Psi)\right. \\ &\qquad \left. - \nabla[\nabla\cdot(\mathbf{A}\Psi)] - \nabla(\mathbf{A}\cdot\nabla\Psi) - \mathbf{A}\nabla^2\Psi\right\}d\mathcal{V} \\ &\quad + \frac{q^2}{2m}\iiint_{\text{all space}}\Psi^*\left\{\mathbf{A}\cdot(\mathbf{A}\nabla\Psi) + \nabla\cdot(\mathbf{A}\mathbf{A}\Psi) + \mathbf{A}\cdot\nabla(\mathbf{A}\Psi)\right. \\ &\qquad \left. - \nabla(A^2\Psi) - \mathbf{A}[\nabla\cdot(\mathbf{A}\Psi)] - \mathbf{A}(\mathbf{A}\cdot\nabla\Psi)\right\}d\mathcal{V} \\ &\quad + q\langle\mathbf{E}\rangle \end{aligned}$$

Expand all of the vectors and operators.

$$\frac{iq\hbar}{2m}\iiint\limits_{\text{all space}}\Psi^*\left\{\left(\sum_{j=1}^{3}\boldsymbol{\delta}_j\frac{\partial}{\partial x_j}\right)\cdot\left[\left(\sum_{k=1}^{3}\boldsymbol{\delta}_k A_k\right)\left(\sum_{l=1}^{3}\boldsymbol{\delta}_l\frac{\partial\Psi}{\partial x_l}\right)\right]+\left(\sum_{j=1}^{3}\boldsymbol{\delta}_j A_j\right)\cdot\left[\left(\sum_{k=1}^{3}\boldsymbol{\delta}_k\frac{\partial}{\partial x_k}\right)\left(\sum_{l=1}^{3}\boldsymbol{\delta}_l\frac{\partial\Psi}{\partial x_l}\right)\right]+\left(\sum_{k=1}^{3}\frac{\partial^2}{\partial x_k^2}\right)\left(\sum_{l=1}^{3}\boldsymbol{\delta}_l A_l\Psi\right)\right.$$

$$\left.-\left(\sum_{j=1}^{3}\boldsymbol{\delta}_j\frac{\partial}{\partial x_j}\right)\left[\left(\sum_{k=1}^{3}\boldsymbol{\delta}_k\frac{\partial}{\partial x_k}\right)\cdot\left(\sum_{l=1}^{3}\boldsymbol{\delta}_l A_l\Psi\right)\right]-\left(\sum_{j=1}^{3}\boldsymbol{\delta}_j\frac{\partial}{\partial x_j}\right)\left[\left(\sum_{k=1}^{3}\boldsymbol{\delta}_k A_k\right)\cdot\left(\sum_{l=1}^{3}\boldsymbol{\delta}_l\frac{\partial\Psi}{\partial x_l}\right)\right]-\left(\sum_{k=1}^{3}\boldsymbol{\delta}_k A_k\right)\left(\sum_{l=1}^{3}\frac{\partial^2\Psi}{\partial x_l^2}\right)\right\}d\mathcal{V}$$

$$+\frac{q^2}{2m}\iiint\limits_{\text{all space}}\Psi^*\left\{\left(\sum_{j=1}^{3}\boldsymbol{\delta}_j A_j\right)\cdot\left(\sum_{k=1}^{3}\boldsymbol{\delta}_k A_k\right)\left(\sum_{l=1}^{3}\boldsymbol{\delta}_l\frac{\partial\Psi}{\partial x_l}\right)+\left(\sum_{j=1}^{3}\boldsymbol{\delta}_j\frac{\partial}{\partial x_j}\right)\cdot\left(\sum_{k=1}^{3}\boldsymbol{\delta}_k A_k\right)\left(\sum_{l=1}^{3}\boldsymbol{\delta}_l A_l\Psi\right)+\left(\sum_{j=1}^{3}\boldsymbol{\delta}_j A_j\right)\cdot\left(\sum_{k=1}^{3}\boldsymbol{\delta}_k\frac{\partial}{\partial x_k}\right)\left(\sum_{l=1}^{3}\boldsymbol{\delta}_l A_l\Psi\right)\right.$$

$$\left.-\left(\sum_{j=1}^{3}\boldsymbol{\delta}_j\frac{\partial}{\partial x_j}\right)\left(\sum_{k=1}^{3}A_k^2\Psi\right)-\left(\sum_{j=1}^{3}\boldsymbol{\delta}_j A_j\right)\left[\left(\sum_{k=1}^{3}\boldsymbol{\delta}_k\frac{\partial}{\partial x_k}\right)\cdot\left(\sum_{l=1}^{3}\boldsymbol{\delta}_l A_l\Psi\right)\right]-\left(\sum_{j=1}^{3}\boldsymbol{\delta}_j A_j\right)\left[\left(\sum_{k=1}^{3}\boldsymbol{\delta}_k A_k\right)\cdot\left(\sum_{l=1}^{3}\boldsymbol{\delta}_l\frac{\partial\Psi}{\partial x_l}\right)\right]\right\}d\mathcal{V}$$

$$+q\langle\mathbf{E}\rangle=m\frac{d\langle\mathbf{v}\rangle}{dt}$$

Continue the simplification.

$$m\frac{d}{dt}\langle\mathbf{v}\rangle = \frac{iq\hbar}{2m}\iiint\limits_{\text{all space}}\Psi^*\left[\left(\sum_{j=1}^{3}\boldsymbol{\delta}_j\frac{\partial}{\partial x_j}\right)\cdot\left(\sum_{k=1}^{3}\sum_{l=1}^{3}\boldsymbol{\delta}_k\boldsymbol{\delta}_l A_k\frac{\partial\Psi}{\partial x_l}\right)+\left(\sum_{j=1}^{3}\boldsymbol{\delta}_j A_j\right)\cdot\left(\sum_{k=1}^{3}\sum_{l=1}^{3}\boldsymbol{\delta}_k\boldsymbol{\delta}_l\frac{\partial^2\Psi}{\partial x_k\partial x_l}\right)+\sum_{k=1}^{3}\sum_{l=1}^{3}\boldsymbol{\delta}_l\frac{\partial^2}{\partial x_k^2}(A_l\Psi)\right.$$

$$\left.-\left(\sum_{j=1}^{3}\boldsymbol{\delta}_j\frac{\partial}{\partial x_j}\right)\sum_{k=1}^{3}\sum_{l=1}^{3}(\boldsymbol{\delta}_k\cdot\boldsymbol{\delta}_l)\frac{\partial}{\partial x_k}(A_l\Psi)-\left(\sum_{j=1}^{3}\boldsymbol{\delta}_j\frac{\partial}{\partial x_j}\right)\sum_{k=1}^{3}\sum_{l=1}^{3}(\boldsymbol{\delta}_k\cdot\boldsymbol{\delta}_l)A_k\frac{\partial\Psi}{\partial x_l}-\sum_{k=1}^{3}\sum_{l=1}^{3}\boldsymbol{\delta}_k A_k\frac{\partial^2\Psi}{\partial x_l^2}\right]d\mathcal{V}$$

$$+\frac{q^2}{2m}\iiint\limits_{\text{all space}}\Psi^*\left[\left(\sum_{j=1}^{3}\boldsymbol{\delta}_j A_j\right)\cdot\left(\sum_{k=1}^{3}\sum_{l=1}^{3}\boldsymbol{\delta}_k\boldsymbol{\delta}_l A_k\frac{\partial\Psi}{\partial x_l}\right)+\left(\sum_{j=1}^{3}\boldsymbol{\delta}_j\frac{\partial}{\partial x_j}\right)\cdot\left(\sum_{k=1}^{3}\sum_{l=1}^{3}\boldsymbol{\delta}_k\boldsymbol{\delta}_l A_k A_l\Psi\right)+\left(\sum_{j=1}^{3}\boldsymbol{\delta}_j A_j\right)\cdot\sum_{k=1}^{3}\sum_{l=1}^{3}\boldsymbol{\delta}_k\boldsymbol{\delta}_l\frac{\partial}{\partial x_k}(A_l\Psi)\right.$$

$$\left.-\sum_{j=1}^{3}\sum_{k=1}^{3}\boldsymbol{\delta}_j\frac{\partial}{\partial x_j}(A_k^2\Psi)-\left(\sum_{j=1}^{3}\boldsymbol{\delta}_j A_j\right)\sum_{k=1}^{3}\sum_{l=1}^{3}(\boldsymbol{\delta}_k\cdot\boldsymbol{\delta}_l)\frac{\partial}{\partial x_k}(A_l\Psi)-\left(\sum_{j=1}^{3}\boldsymbol{\delta}_j A_j\right)\sum_{k=1}^{3}\sum_{l=1}^{3}(\boldsymbol{\delta}_k\cdot\boldsymbol{\delta}_l)A_k\frac{\partial\Psi}{\partial x_l}\right]d\mathcal{V}$$

$$+q\langle\mathbf{E}\rangle$$

Continue the simplification.

$$\begin{aligned}
m\frac{d\langle \mathbf{v}\rangle}{dt} &= \frac{iq\hbar}{2m}\iiint\limits_{\text{all space}} \Psi^* \left[\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}(\boldsymbol{\delta}_j\cdot\boldsymbol{\delta}_k)\boldsymbol{\delta}_l\frac{\partial}{\partial x_j}\left(A_k\frac{\partial \Psi}{\partial x_l}\right) + \sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}(\boldsymbol{\delta}_j\cdot\boldsymbol{\delta}_k)\boldsymbol{\delta}_l A_j\frac{\partial^2\Psi}{\partial x_k\partial x_l} + \sum_{k=1}^{3}\sum_{l=1}^{3}\boldsymbol{\delta}_l\frac{\partial^2}{\partial x_k^2}(A_l\Psi)\right.\\
&\qquad \left. - \left(\sum_{j=1}^{3}\boldsymbol{\delta}_j\frac{\partial}{\partial x_j}\right)\sum_{k=1}^{3}\sum_{l=1}^{3}\delta_{kl}\frac{\partial}{\partial x_k}(A_l\Psi) - \left(\sum_{j=1}^{3}\boldsymbol{\delta}_j\frac{\partial}{\partial x_j}\right)\sum_{k=1}^{3}\sum_{l=1}^{3}\delta_{kl}A_k\frac{\partial\Psi}{\partial x_l} - \sum_{k=1}^{3}\sum_{l=1}^{3}\boldsymbol{\delta}_k A_k\frac{\partial^2\Psi}{\partial x_l^2}\right] d\mathcal{V}\\
&\quad + \frac{q^2}{2m}\iiint\limits_{\text{all space}} \Psi^* \left[\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}(\boldsymbol{\delta}_j\cdot\boldsymbol{\delta}_k)\boldsymbol{\delta}_l A_j A_k\frac{\partial\Psi}{\partial x_l} + \sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}(\boldsymbol{\delta}_j\cdot\boldsymbol{\delta}_k)\boldsymbol{\delta}_l\frac{\partial}{\partial x_j}(A_k A_l\Psi) + \sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}(\boldsymbol{\delta}_j\cdot\boldsymbol{\delta}_k)\boldsymbol{\delta}_l A_j\frac{\partial}{\partial x_k}(A_l\Psi)\right.\\
&\qquad \left. - \sum_{j=1}^{3}\sum_{k=1}^{3}\boldsymbol{\delta}_j\frac{\partial}{\partial x_j}(A_k^2\Psi) - \left(\sum_{j=1}^{3}\boldsymbol{\delta}_j A_j\right)\sum_{k=1}^{3}\sum_{l=1}^{3}\delta_{kl}\frac{\partial}{\partial x_k}(A_l\Psi) - \left(\sum_{j=1}^{3}\boldsymbol{\delta}_j A_j\right)\sum_{k=1}^{3}\sum_{l=1}^{3}\delta_{kl}A_k\frac{\partial\Psi}{\partial x_l}\right] d\mathcal{V}\\
&\quad + q\langle \mathbf{E}\rangle
\end{aligned}$$

Continue the simplification.

$$m\frac{d\langle\mathbf{v}\rangle}{dt} = \frac{iq\hbar}{2m}\iiint\limits_{\text{all space}}\Psi^*\left[\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\delta_{jk}\boldsymbol{\delta}_l\frac{\partial}{\partial x_j}\left(A_k\frac{\partial\Psi}{\partial x_l}\right)+\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\delta_{jk}\boldsymbol{\delta}_l A_j\frac{\partial^2\Psi}{\partial x_k\partial x_l}+\sum_{k=1}^{3}\sum_{l=1}^{3}\boldsymbol{\delta}_l\frac{\partial^2}{\partial x_k^2}(A_l\Psi)\right.$$

$$\left.-\left(\sum_{j=1}^{3}\boldsymbol{\delta}_j\frac{\partial}{\partial x_j}\right)\sum_{k=1}^{3}\frac{\partial}{\partial x_k}(A_k\Psi)-\left(\sum_{j=1}^{3}\boldsymbol{\delta}_j\frac{\partial}{\partial x_j}\right)\sum_{k=1}^{3}A_k\frac{\partial\Psi}{\partial x_k}-\sum_{k=1}^{3}\sum_{l=1}^{3}\boldsymbol{\delta}_k A_k\frac{\partial^2\Psi}{\partial x_l^2}\right]d\mathcal{V}$$

$$+\frac{q^2}{2m}\iiint\limits_{\text{all space}}\Psi^*\left[\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\delta_{jk}\boldsymbol{\delta}_l A_j A_k\frac{\partial\Psi}{\partial x_l}+\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\delta_{jk}\boldsymbol{\delta}_l\frac{\partial}{\partial x_j}(A_k A_l\Psi)+\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\delta_{jk}\boldsymbol{\delta}_l A_j\frac{\partial}{\partial x_k}(A_l\Psi)\right.$$

$$\left.-\sum_{j=1}^{3}\sum_{k=1}^{3}\boldsymbol{\delta}_j\frac{\partial}{\partial x_j}(A_k^2\Psi)-\left(\sum_{j=1}^{3}\boldsymbol{\delta}_j A_j\right)\sum_{k=1}^{3}\frac{\partial}{\partial x_k}(A_k\Psi)-\left(\sum_{j=1}^{3}\boldsymbol{\delta}_j A_j\right)\sum_{k=1}^{3}A_k\frac{\partial\Psi}{\partial x_k}\right]d\mathcal{V}$$

$$+q\langle\mathbf{E}\rangle$$

Continue the simplification.

$$m\frac{d\langle\mathbf{v}\rangle}{dt} = \frac{iq\hbar}{2m}\iiint\limits_{\text{all space}} \Psi^* \left[\sum_{k=1}^{3}\sum_{l=1}^{3}\boldsymbol{\delta}_l\frac{\partial}{\partial x_k}\left(A_k\frac{\partial\Psi}{\partial x_l}\right) + \sum_{k=1}^{3}\sum_{l=1}^{3}\boldsymbol{\delta}_l A_k\frac{\partial^2\Psi}{\partial x_k\partial x_l} + \sum_{k=1}^{3}\sum_{l=1}^{3}\boldsymbol{\delta}_l\frac{\partial^2}{\partial x_k^2}(A_l\Psi)\right.$$

$$\left. - \sum_{j=1}^{3}\sum_{k=1}^{3}\boldsymbol{\delta}_j\frac{\partial^2}{\partial x_j\partial x_k}(A_k\Psi) - \sum_{j=1}^{3}\sum_{k=1}^{3}\boldsymbol{\delta}_j\frac{\partial}{\partial x_j}\left(A_k\frac{\partial\Psi}{\partial x_k}\right) - \sum_{k=1}^{3}\sum_{l=1}^{3}\boldsymbol{\delta}_k A_k\frac{\partial^2\Psi}{\partial x_l^2}\right] d\mathcal{V}$$

$$+ \frac{q^2}{2m}\iiint\limits_{\text{all space}} \Psi^* \left[\sum_{k=1}^{3}\sum_{l=1}^{3}\boldsymbol{\delta}_l A_k A_k\frac{\partial\Psi}{\partial x_l} + \sum_{k=1}^{3}\sum_{l=1}^{3}\boldsymbol{\delta}_l\frac{\partial}{\partial x_k}(A_k A_l\Psi) + \sum_{k=1}^{3}\sum_{l=1}^{3}\boldsymbol{\delta}_l A_k\frac{\partial}{\partial x_k}(A_l\Psi)\right.$$

$$\left. - \sum_{j=1}^{3}\sum_{k=1}^{3}\boldsymbol{\delta}_j\frac{\partial}{\partial x_j}(A_k^2\Psi) - \sum_{j=1}^{3}\sum_{k=1}^{3}\boldsymbol{\delta}_j A_j\frac{\partial}{\partial x_k}(A_k\Psi) - \sum_{j=1}^{3}\sum_{k=1}^{3}\boldsymbol{\delta}_j A_j A_k\frac{\partial\Psi}{\partial x_k}\right] d\mathcal{V}$$

$$+ q\langle\mathbf{E}\rangle$$

Relabel the dummy index $j$ as $l$ so that the unit vector is the same in each term. This allows the entire expression to be factored.

$$\begin{aligned}
m\frac{d\langle \mathbf{v}\rangle}{dt} &= \frac{iq\hbar}{2m}\iiint_{\text{all space}} \Psi^*\left[\sum_{k=1}^{3}\sum_{l=1}^{3}\boldsymbol{\delta}_l\frac{\partial}{\partial x_k}\left(A_k\frac{\partial\Psi}{\partial x_l}\right)+\sum_{k=1}^{3}\sum_{l=1}^{3}\boldsymbol{\delta}_l A_k\frac{\partial^2\Psi}{\partial x_k\partial x_l}+\sum_{k=1}^{3}\sum_{l=1}^{3}\boldsymbol{\delta}_l\frac{\partial^2}{\partial x_k^2}(A_l\Psi)\right.\\
&\qquad\left.-\sum_{l=1}^{3}\sum_{k=1}^{3}\boldsymbol{\delta}_l\frac{\partial^2}{\partial x_l\partial x_k}(A_k\Psi)-\sum_{l=1}^{3}\sum_{k=1}^{3}\boldsymbol{\delta}_l\frac{\partial}{\partial x_l}\left(A_k\frac{\partial\Psi}{\partial x_k}\right)-\sum_{l=1}^{3}\sum_{k=1}^{3}\boldsymbol{\delta}_l A_l\frac{\partial^2\Psi}{\partial x_k^2}\right]d\mathcal{V}\\
&\quad+\frac{q^2}{2m}\iiint_{\text{all space}}\Psi^*\left[\sum_{k=1}^{3}\sum_{l=1}^{3}\boldsymbol{\delta}_l A_k A_k\frac{\partial\Psi}{\partial x_l}+\sum_{k=1}^{3}\sum_{l=1}^{3}\boldsymbol{\delta}_l\frac{\partial}{\partial x_k}(A_kA_l\Psi)+\sum_{k=1}^{3}\sum_{l=1}^{3}\boldsymbol{\delta}_l A_k\frac{\partial}{\partial x_k}(A_l\Psi)\right.\\
&\qquad\left.-\sum_{l=1}^{3}\sum_{k=1}^{3}\boldsymbol{\delta}_l\frac{\partial}{\partial x_l}(A_k^2\Psi)-\sum_{l=1}^{3}\sum_{k=1}^{3}\boldsymbol{\delta}_l A_l\frac{\partial}{\partial x_k}(A_k\Psi)-\sum_{l=1}^{3}\sum_{k=1}^{3}\boldsymbol{\delta}_l A_lA_k\frac{\partial\Psi}{\partial x_k}\right]d\mathcal{V}\\
&\quad+q\langle\mathbf{E}\rangle\\
&= \frac{iq\hbar}{2m}\iiint_{\text{all space}}\Psi^*\sum_{k=1}^{3}\sum_{l=1}^{3}\boldsymbol{\delta}_l\left[\frac{\partial}{\partial x_k}\left(A_k\frac{\partial\Psi}{\partial x_l}\right)+A_k\frac{\partial^2\Psi}{\partial x_k\partial x_l}+\frac{\partial^2}{\partial x_k^2}(A_l\Psi)-\frac{\partial^2}{\partial x_l\partial x_k}(A_k\Psi)-\frac{\partial}{\partial x_l}\left(A_k\frac{\partial\Psi}{\partial x_k}\right)-A_l\frac{\partial^2\Psi}{\partial x_k^2}\right]d\mathcal{V}\\
&\quad+\frac{q^2}{2m}\iiint_{\text{all space}}\Psi^*\sum_{k=1}^{3}\sum_{l=1}^{3}\boldsymbol{\delta}_l\left[A_k^2\frac{\partial\Psi}{\partial x_l}+\frac{\partial}{\partial x_k}(A_kA_l\Psi)+A_k\frac{\partial}{\partial x_k}(A_l\Psi)-\frac{\partial}{\partial x_l}(A_k^2\Psi)-A_l\frac{\partial}{\partial x_k}(A_k\Psi)-A_lA_k\frac{\partial\Psi}{\partial x_k}\right]d\mathcal{V}\\
&\quad+q\langle\mathbf{E}\rangle
\end{aligned}$$

Take derivatives using the product rule and then simplify the result.

$$
\begin{aligned}
m\frac{d\langle \mathbf{v}\rangle}{dt} &= \frac{iq\hbar}{2m}\iiint\limits_{\text{all space}} \Psi^* \sum_{k=1}^{3}\sum_{l=1}^{3} \boldsymbol{\delta}_l \left[\left(\frac{\partial A_k}{\partial x_k}\frac{\partial \Psi}{\partial x_l} + A_k\frac{\partial^2 \Psi}{\partial x_k \partial x_l}\right) + A_k\frac{\partial^2\Psi}{\partial x_k\partial x_l} + \frac{\partial}{\partial x_k}\left(\frac{\partial A_l}{\partial x_k}\Psi + A_l\frac{\partial \Psi}{\partial x_k}\right)\right. \\
&\qquad \left. -\frac{\partial}{\partial x_l}\left(\frac{\partial A_k}{\partial x_k}\Psi + A_k\frac{\partial\Psi}{\partial x_k}\right) - \left(\frac{\partial A_k}{\partial x_l}\frac{\partial\Psi}{\partial x_k} + A_k\frac{\partial^2\Psi}{\partial x_l\partial x_k}\right) - A_l\frac{\partial^2\Psi}{\partial x_k^2}\right] d\mathcal{V} \\
&\quad + \frac{q^2}{2m}\iiint\limits_{\text{all space}} \Psi^* \sum_{k=1}^{3}\sum_{l=1}^{3} \boldsymbol{\delta}_l \left[A_k^2\frac{\partial\Psi}{\partial x_l} + \left(\frac{\partial A_k}{\partial x_k}A_l\Psi + A_k\frac{\partial A_l}{\partial x_k}\Psi + A_kA_l\frac{\partial\Psi}{\partial x_k}\right) + A_k\left(\frac{\partial A_l}{\partial x_k}\Psi + A_l\frac{\partial\Psi}{\partial x_k}\right)\right. \\
&\qquad \left. -\left(2A_k\frac{\partial A_k}{\partial x_l}\Psi + A_k^2\frac{\partial\Psi}{\partial x_l}\right) - A_l\left(\frac{\partial A_k}{\partial x_k}\Psi + A_k\frac{\partial\Psi}{\partial x_k}\right) - A_lA_k\frac{\partial\Psi}{\partial x_k}\right] d\mathcal{V} \\
&\quad + q\langle \mathbf{E}\rangle \\
&= \frac{iq\hbar}{2m}\iiint\limits_{\text{all space}} \Psi^* \sum_{k=1}^{3}\sum_{l=1}^{3} \boldsymbol{\delta}_l \left(\cancel{\frac{\partial A_k}{\partial x_k}\frac{\partial\Psi}{\partial x_l}} + \cancel{A_k\frac{\partial^2\Psi}{\partial x_k\partial x_l}} + \cancel{A_k\frac{\partial^2\Psi}{\partial x_k\partial x_l}} + \frac{\partial^2 A_l}{\partial x_k^2}\Psi + \frac{\partial A_l}{\partial x_k}\frac{\partial\Psi}{\partial x_k} + \frac{\partial A_l}{\partial x_k}\frac{\partial\Psi}{\partial x_k} + \cancel{A_l\frac{\partial^2\Psi}{\partial x_k^2}}\right. \\
&\qquad \left. -\frac{\partial^2 A_k}{\partial x_l\partial x_k}\Psi - \cancel{\frac{\partial A_k}{\partial x_k}\frac{\partial\Psi}{\partial x_l}} - \frac{\partial A_k}{\partial x_l}\frac{\partial\Psi}{\partial x_k} - \cancel{A_k\frac{\partial^2\Psi}{\partial x_l\partial x_k}} - \frac{\partial A_k}{\partial x_l}\frac{\partial\Psi}{\partial x_k} - \cancel{A_k\frac{\partial^2\Psi}{\partial x_l\partial x_k}} - \cancel{A_l\frac{\partial^2\Psi}{\partial x_k^2}}\right) d\mathcal{V} \\
&\quad + \frac{q^2}{2m}\iiint\limits_{\text{all space}} \Psi^* \sum_{k=1}^{3}\sum_{l=1}^{3} \boldsymbol{\delta}_l \left(\cancel{A_k^2\frac{\partial\Psi}{\partial x_l}} + \cancel{\frac{\partial A_k}{\partial x_k}A_l\Psi} + A_k\frac{\partial A_l}{\partial x_k}\Psi + \cancel{A_kA_l\frac{\partial\Psi}{\partial x_k}} + A_k\frac{\partial A_l}{\partial x_k}\Psi + \cancel{A_kA_l\frac{\partial\Psi}{\partial x_k}}\right. \\
&\qquad \left. -2A_k\frac{\partial A_k}{\partial x_l}\Psi - \cancel{A_k^2\frac{\partial\Psi}{\partial x_l}} - \cancel{A_l\frac{\partial A_k}{\partial x_k}\Psi} - \cancel{A_kA_l\frac{\partial\Psi}{\partial x_k}} - \cancel{A_lA_k\frac{\partial\Psi}{\partial x_k}}\right) d\mathcal{V} \\
&\quad + q\langle \mathbf{E}\rangle
\end{aligned}
$$

Continue the simplification, noting that $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$.

$$m\frac{d\langle \mathbf{v}\rangle}{dt} = \frac{iq\hbar}{2m}\iiint\limits_{\text{all space}} \Psi^* \sum_{k=1}^3 \sum_{l=1}^3 \boldsymbol{\delta}_l \left(\frac{\partial^2 A_l}{\partial x_k^2}\Psi + 2\frac{\partial A_l}{\partial x_k}\frac{\partial \Psi}{\partial x_k} - \frac{\partial^2 A_k}{\partial x_l \partial x_k}\Psi - 2\frac{\partial A_k}{\partial x_l}\frac{\partial \Psi}{\partial x_k}\right) d\mathcal{V}$$

$$+ \frac{q^2}{2m}\iiint\limits_{\text{all space}} \Psi^* \sum_{k=1}^3 \sum_{l=1}^3 \boldsymbol{\delta}_l \left(2A_k \frac{\partial A_l}{\partial x_k}\Psi - 2A_k\frac{\partial A_k}{\partial x_l}\Psi\right) d\mathcal{V}$$

$$+ q\langle \mathbf{E}\rangle$$

$$= \frac{iq\hbar}{2m}\iiint\limits_{\text{all space}} \Psi^* \left[\sum_{k=1}^3 \frac{\partial^2}{\partial x_k^2}\left(\sum_{l=1}^3 \boldsymbol{\delta}_l A_l\right)\Psi + 2\sum_{k=1}^3\sum_{l=1}^3 \boldsymbol{\delta}_l \left(\frac{\partial A_l}{\partial x_k} - \frac{\partial A_k}{\partial x_l}\right)\frac{\partial \Psi}{\partial x_k} - \sum_{l=1}^3 \boldsymbol{\delta}_l \frac{\partial}{\partial x_l}\left(\sum_{k=1}^3 \frac{\partial A_k}{\partial x_k}\right)\Psi\right] d\mathcal{V}$$

$$+ \frac{q^2}{m}\iiint\limits_{\text{all space}} \Psi^* \sum_{k=1}^3 \sum_{l=1}^3 \boldsymbol{\delta}_l A_k \left(\frac{\partial A_l}{\partial x_k} - \frac{\partial A_k}{\partial x_l}\right)\Psi \, d\mathcal{V}$$

$$+ q\langle \mathbf{E}\rangle$$

$$= \frac{iq\hbar}{2m}\iiint\limits_{\text{all space}} \Psi^* \left[(\nabla^2 \mathbf{A})\Psi + 2\left(\sum_{k=1}^3\sum_{l=1}^3 \boldsymbol{\delta}_l \frac{\partial A_l}{\partial x_k}\frac{\partial \Psi}{\partial x_k} - \sum_{k=1}^3\sum_{l=1}^3 \boldsymbol{\delta}_l \frac{\partial A_k}{\partial x_l}\frac{\partial \Psi}{\partial x_k}\right) - \nabla(\nabla \cdot \mathbf{A})\Psi\right] d\mathcal{V}$$

$$+ \frac{q^2}{m}\iiint\limits_{\text{all space}} \Psi^* \left(\sum_{k=1}^3\sum_{l=1}^3 \boldsymbol{\delta}_l A_k \frac{\partial A_l}{\partial x_k} - \sum_{k=1}^3\sum_{l=1}^3 \boldsymbol{\delta}_l A_k \frac{\partial A_k}{\partial x_l}\right)\Psi \, d\mathcal{V}$$

$$+ q\langle \mathbf{E}\rangle$$

$$= \frac{iq\hbar}{2m}\iiint\limits_{\text{all space}} \Psi^* \left\{-[\nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}]\Psi + 2\left(\sum_{k=1}^3\sum_{l=1}^3 \boldsymbol{\delta}_k \frac{\partial A_k}{\partial x_l}\frac{\partial \Psi}{\partial x_l} - \sum_{k=1}^3\sum_{l=1}^3 \boldsymbol{\delta}_l \frac{\partial A_k}{\partial x_l}\frac{\partial \Psi}{\partial x_k}\right)\right\} d\mathcal{V}$$

$$+ \frac{q^2}{m}\iiint\limits_{\text{all space}} \Psi^* \left(\sum_{k=1}^3\sum_{l=1}^3 \boldsymbol{\delta}_k A_l \frac{\partial A_k}{\partial x_l} - \sum_{k=1}^3\sum_{l=1}^3 \boldsymbol{\delta}_l A_k \frac{\partial A_k}{\partial x_l}\right)\Psi \, d\mathcal{V}$$

$$+ q\langle \mathbf{E}\rangle$$

$$= \frac{iq\hbar}{2m}\iiint\limits_{\text{all space}} \Psi^* \left\{-[\nabla \times (\nabla \times \mathbf{A})]\Psi + 2\left(\sum_{b=1}^3\sum_{k=1}^3\sum_{l=1}^3 \delta_{bl}\boldsymbol{\delta}_k \frac{\partial A_k}{\partial x_l}\frac{\partial \Psi}{\partial x_b} - \sum_{b=1}^3\sum_{k=1}^3\sum_{l=1}^3 \delta_{bk}\boldsymbol{\delta}_l \frac{\partial A_k}{\partial x_l}\frac{\partial \Psi}{\partial x_b}\right)\right\} d\mathcal{V}$$

$$+ \frac{q^2}{m}\iiint\limits_{\text{all space}} \Psi^* \left(\sum_{b=1}^3\sum_{k=1}^3\sum_{l=1}^3 \delta_{bl}\boldsymbol{\delta}_k A_b \frac{\partial A_k}{\partial x_l} - \sum_{b=1}^3\sum_{k=1}^3\sum_{l=1}^3 \delta_{bk}\boldsymbol{\delta}_l A_b \frac{\partial A_k}{\partial x_l}\right)\Psi \, d\mathcal{V}$$

$$+ q\langle \mathbf{E}\rangle$$

Continue the simplification.

$$m\frac{d\langle \mathbf{v}\rangle}{dt} = \frac{iq\hbar}{2m}\iiint\limits_{\text{all space}} \Psi^* \left\{-[\nabla\times(\nabla\times\mathbf{A})]\Psi + 2\left(\sum_{a=1}^{3}\sum_{b=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\delta_{ak}\delta_{bl}\boldsymbol{\delta}_a\frac{\partial A_k}{\partial x_l}\frac{\partial\Psi}{\partial x_b} - \sum_{a=1}^{3}\sum_{b=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\delta_{al}\delta_{bk}\boldsymbol{\delta}_a\frac{\partial A_k}{\partial x_l}\frac{\partial\Psi}{\partial x_b}\right)\right\} d\mathcal{V}$$

$$+ \frac{q^2}{m}\iiint\limits_{\text{all space}} \Psi^* \left(\sum_{a=1}^{3}\sum_{b=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\delta_{ak}\delta_{bl}\boldsymbol{\delta}_a A_b\frac{\partial A_k}{\partial x_l} - \sum_{a=1}^{3}\sum_{b=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\delta_{al}\delta_{bk}\boldsymbol{\delta}_a A_b\frac{\partial A_k}{\partial x_l}\right)\Psi\, d\mathcal{V}$$

$$+ q\langle\mathbf{E}\rangle$$

$$= \frac{iq\hbar}{2m}\iiint\limits_{\text{all space}} \Psi^* \left\{-[\nabla\times(\nabla\times\mathbf{A})]\Psi + 2\sum_{a=1}^{3}\sum_{b=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}(\delta_{ak}\delta_{bl}-\delta_{al}\delta_{bk})\boldsymbol{\delta}_a\frac{\partial A_k}{\partial x_l}\frac{\partial\Psi}{\partial x_b}\right\} d\mathcal{V}$$

$$+ \frac{q^2}{m}\iiint\limits_{\text{all space}} \Psi^* \sum_{a=1}^{3}\sum_{b=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}(\delta_{ak}\delta_{bl}-\delta_{al}\delta_{bk})\boldsymbol{\delta}_a A_b\frac{\partial A_k}{\partial x_l}\Psi\, d\mathcal{V}$$

$$+ q\langle\mathbf{E}\rangle$$

$$= \frac{iq\hbar}{2m}\iiint\limits_{\text{all space}} \Psi^* \left\{-[\nabla\times(\nabla\times\mathbf{A})]\Psi + 2\sum_{a=1}^{3}\sum_{b=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\left(\sum_{j=1}^{3}\varepsilon_{jab}\varepsilon_{jkl}\right)\boldsymbol{\delta}_a\frac{\partial A_k}{\partial x_l}\frac{\partial\Psi}{\partial x_b}\right\} d\mathcal{V}$$

$$+ \frac{q^2}{m}\iiint\limits_{\text{all space}} \Psi^* \sum_{a=1}^{3}\sum_{b=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\left(\sum_{j=1}^{3}\varepsilon_{jab}\varepsilon_{jkl}\right)\boldsymbol{\delta}_a A_b\frac{\partial A_k}{\partial x_l}\Psi\, d\mathcal{V}$$

$$+ q\langle\mathbf{E}\rangle$$

Continue the simplification.

$$m\frac{d\langle \mathbf{v}\rangle}{dt} = \frac{iq\hbar}{2m}\iiint\limits_{\text{all space}} \Psi^* \left\{-[\nabla\times(\nabla\times\mathbf{A})]\Psi + 2\sum_{a=1}^{3}\sum_{b=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\varepsilon_{jab}\varepsilon_{jkl}\boldsymbol{\delta}_a\frac{\partial A_k}{\partial x_l}\frac{\partial\Psi}{\partial x_b}\right\} d\mathcal{V}$$

$$+\frac{q^2}{m}\iiint\limits_{\text{all space}} \Psi^* \sum_{a=1}^{3}\sum_{b=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\varepsilon_{jab}\varepsilon_{jkl}\boldsymbol{\delta}_a A_b\frac{\partial A_k}{\partial x_l}\Psi\, d\mathcal{V}$$

$$+ q\langle\mathbf{E}\rangle$$

$$= \frac{iq\hbar}{2m}\iiint\limits_{\text{all space}} \Psi^* \left\{-[\nabla\times(\nabla\times\mathbf{A})]\Psi + 2\sum_{a=1}^{3}\sum_{b=1}^{3}\sum_{j=1}^{3}\varepsilon_{jab}\boldsymbol{\delta}_a\left(\sum_{k=1}^{3}\sum_{l=1}^{3}\varepsilon_{jkl}\frac{\partial A_k}{\partial x_l}\right)\frac{\partial\Psi}{\partial x_b}\right\} d\mathcal{V}$$

$$+\frac{q^2}{m}\iiint\limits_{\text{all space}} \Psi^* \sum_{a=1}^{3}\sum_{b=1}^{3}\sum_{j=1}^{3}\varepsilon_{jab}\boldsymbol{\delta}_a A_b\left(\sum_{k=1}^{3}\sum_{l=1}^{3}\varepsilon_{jkl}\frac{\partial A_k}{\partial x_l}\right)\Psi\, d\mathcal{V}$$

$$+ q\langle\mathbf{E}\rangle$$

$$= \frac{iq\hbar}{2m}\iiint\limits_{\text{all space}} \Psi^* \left\{-[\nabla\times(\nabla\times\mathbf{A})]\Psi + 2\sum_{a=1}^{3}\sum_{b=1}^{3}\sum_{j=1}^{3}\varepsilon_{jab}\boldsymbol{\delta}_a\left(-\sum_{k=1}^{3}\sum_{l=1}^{3}\varepsilon_{jlk}\frac{\partial A_k}{\partial x_l}\right)\frac{\partial\Psi}{\partial x_b}\right\} d\mathcal{V}$$

$$+\frac{q^2}{m}\iiint\limits_{\text{all space}} \Psi^* \sum_{a=1}^{3}\sum_{b=1}^{3}\sum_{j=1}^{3}\varepsilon_{jab}\boldsymbol{\delta}_a A_b\left(-\sum_{k=1}^{3}\sum_{l=1}^{3}\varepsilon_{jlk}\frac{\partial A_k}{\partial x_l}\right)\Psi\, d\mathcal{V}$$

$$+ q\langle\mathbf{E}\rangle$$

$$= \frac{iq\hbar}{2m}\iiint\limits_{\text{all space}} \Psi^* \left\{-[\nabla\times(\nabla\times\mathbf{A})]\Psi + 2\sum_{a=1}^{3}\sum_{b=1}^{3}\sum_{j=1}^{3}(-\varepsilon_{jab})\boldsymbol{\delta}_a(\nabla\times\mathbf{A})_j\frac{\partial\Psi}{\partial x_b}\right\} d\mathcal{V}$$

$$-\frac{q^2}{m}\iiint\limits_{\text{all space}} \Psi^* \sum_{a=1}^{3}\sum_{b=1}^{3}\sum_{j=1}^{3}\varepsilon_{jab}\boldsymbol{\delta}_a A_b(\nabla\times\mathbf{A})_j\Psi\, d\mathcal{V}$$

$$+ q\langle\mathbf{E}\rangle$$

Continue the simplification.

$$m\frac{d\langle \mathbf{v}\rangle}{dt} = \frac{iq\hbar}{2m}\iiint\limits_{\text{all space}} \Psi^* \left\{-[\nabla\times(\nabla\times\mathbf{A})]\Psi + 2\left[\sum_{a=1}^{3}\sum_{b=1}^{3}\sum_{j=1}^{3}\varepsilon_{ajb}\boldsymbol{\delta}_a(\nabla\times\mathbf{A})_j\frac{\partial\Psi}{\partial x_b}\right]\right\} d\mathcal{V}$$

$$- \frac{q^2}{m}\iiint\limits_{\text{all space}} \Psi^* \left[\sum_{a=1}^{3}\sum_{b=1}^{3}\sum_{j=1}^{3}\varepsilon_{abj}\boldsymbol{\delta}_a A_b(\nabla\times\mathbf{A})_j\right]\Psi\, d\mathcal{V}$$

$$+ q\langle\mathbf{E}\rangle$$

$$= \frac{iq\hbar}{2m}\iiint\limits_{\text{all space}} \Psi^* \{-[\nabla\times(\nabla\times\mathbf{A})]\Psi + 2[(\nabla\times\mathbf{A})\times(\nabla\Psi)]\}\, d\mathcal{V}$$

$$- \frac{q^2}{m}\iiint\limits_{\text{all space}} \Psi^*[\mathbf{A}\times(\nabla\times\mathbf{A})]\Psi\, d\mathcal{V}$$

$$+ q\langle\mathbf{E}\rangle$$

$$= \frac{q}{2m}\iiint\limits_{\text{all space}} \Psi^* \{[(-i\hbar\nabla)\times(\nabla\times\mathbf{A})]\Psi - 2[(\nabla\times\mathbf{A})\times(-i\hbar\nabla\Psi)]\}\, d\mathcal{V}$$

$$- \frac{q^2}{m}\iiint\limits_{\text{all space}} \Psi^*[\mathbf{A}\times(\nabla\times\mathbf{A})]\Psi\, d\mathcal{V}$$

$$+ q\langle\mathbf{E}\rangle$$

$$= \frac{q}{2m}\iiint\limits_{\text{all space}} \Psi^* [(\mathbf{p}\times\mathbf{B})\Psi - 2(\mathbf{B}\times\mathbf{p})\Psi]\, d\mathcal{V}$$

$$- \frac{q^2}{m}\iiint\limits_{\text{all space}} \Psi^*(\mathbf{A}\times\mathbf{B})\Psi\, d\mathcal{V}$$

$$+ q\langle\mathbf{E}\rangle$$

$$= q\langle\mathbf{E}\rangle + \frac{q}{2m}\iiint\limits_{\text{all space}} \Psi^*(\mathbf{p}\times\mathbf{B} - 2\mathbf{B}\times\mathbf{p})\Psi\, d\mathcal{V} - \frac{q^2}{m}\iiint\limits_{\text{all space}} \Psi^*(\mathbf{A}\times\mathbf{B})\Psi\, d\mathcal{V}$$

$$= q\langle\mathbf{E}\rangle + \frac{q}{2m}\langle\Psi\,|\,\mathbf{p}\times\mathbf{B} - 2\mathbf{B}\times\mathbf{p}\,|\,\Psi\rangle - \frac{q^2}{m}\langle\Psi\,|\,\mathbf{A}\times\mathbf{B}\,|\,\Psi\rangle$$

Therefore,

$$m\frac{d\langle\mathbf{v}\rangle}{dt} = q\langle\mathbf{E}\rangle + \frac{q}{2m}\langle\mathbf{p}\times\mathbf{B} - 2\mathbf{B}\times\mathbf{p}\rangle - \frac{q^2}{m}\langle\mathbf{A}\times\mathbf{B}\rangle.$$

## Part (c)

Begin with the result of part (b).

$$m\frac{d\langle\mathbf{v}\rangle}{dt} = q\langle\mathbf{E}\rangle + \frac{q}{2m}\langle\mathbf{p}\times\mathbf{B} - 2\mathbf{B}\times\mathbf{p}\rangle - \frac{q^2}{m}\langle\mathbf{A}\times\mathbf{B}\rangle$$

$$= q\langle\Psi\,|\,\mathbf{E}\,|\,\Psi\rangle + \frac{q}{2m}\langle\Psi\,|\,\mathbf{p}\times\mathbf{B} - 2\mathbf{B}\times\mathbf{p}\,|\,\Psi\rangle - \frac{q^2}{m}\langle\Psi\,|\,\mathbf{A}\times\mathbf{B}\,|\,\Psi\rangle$$

$$= \frac{q}{2m}\iiint_{\text{all space}} \Psi^*(\mathbf{p}\times\mathbf{B} - 2\mathbf{B}\times\mathbf{p})\Psi\,d\mathcal{V}$$

$$- \frac{q^2}{m}\iiint_{\text{all space}} \Psi^*(\mathbf{A}\times\mathbf{B})\Psi\,d\mathcal{V}$$

$$+ q\iiint_{\text{all space}} \Psi^*\mathbf{E}\Psi\,d\mathcal{V}$$

$$= \frac{q}{2m}\iiint_{\text{all space}} \Psi^*[(-i\hbar\nabla)\times\mathbf{B} - 2\mathbf{B}\times(-i\hbar\nabla)]\Psi\,d\mathcal{V}$$

$$- \frac{q^2}{m}\iiint_{\text{all space}} \Psi^*(\mathbf{A}\times\mathbf{B})\Psi\,d\mathcal{V}$$

$$+ q\iiint_{\text{all space}} \Psi^*\mathbf{E}\Psi\,d\mathcal{V}$$

$$= \frac{iq\hbar}{2m}\iiint_{\text{all space}} \Psi^*[-(\nabla\times\mathbf{B})\Psi + 2\mathbf{B}\times(\nabla\Psi)]\,d\mathcal{V}$$

$$- \frac{q^2}{m}\iiint_{\text{all space}} \Psi^*(\mathbf{A}\times\mathbf{B})\Psi\,d\mathcal{V}$$

$$+ q\iiint_{\text{all space}} \Psi^*\mathbf{E}\Psi\,d\mathcal{V}$$

By assuming that **E** and **B** are uniform inside a wave packet and zero outside of it, they can be treated as constants.

$$m\frac{d\langle\mathbf{v}\rangle}{dt} = \frac{iq\hbar}{2m}\iiint_{\text{all space}} \Psi^*[2\mathbf{B}\times(\nabla\Psi) - (\nabla\times\mathbf{B})\Psi]\,d\mathcal{V}$$

$$-\frac{q^2}{m}\iiint_{\text{all space}} \Psi^*(\mathbf{A}\times\mathbf{B})\Psi\,d\mathcal{V}$$

$$+q\iiint_{\text{all space}} \Psi^*\mathbf{E}\Psi\,d\mathcal{V}$$

$$= \frac{iq\hbar}{2m}\iiint_{\text{all space}} \Psi^*\left(\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\varepsilon_{jkl}\boldsymbol{\delta}_j 2B_k\frac{\partial\Psi}{\partial x_l} - \sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\varepsilon_{jkl}\boldsymbol{\delta}_j\overbrace{\frac{\partial B_l}{\partial x_k}}^{=0}\Psi\right)d\mathcal{V}$$

$$-\frac{q^2}{m}\iiint_{\text{all space}} \Psi^*\left(\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\varepsilon_{jkl}\boldsymbol{\delta}_j A_k B_l\right)\Psi\,d\mathcal{V}$$

$$+q\iiint_{\text{all space}} \Psi^*\mathbf{E}\Psi\,d\mathcal{V}$$

$$= \frac{iq\hbar}{2m}\left(2\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\varepsilon_{jkl}\boldsymbol{\delta}_j B_k\iiint_{\text{all space}} \Psi^*\frac{\partial\Psi}{\partial x_l}\,d\mathcal{V} - 0\right)$$

$$-\frac{q^2}{m}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\varepsilon_{jkl}\boldsymbol{\delta}_j B_l\iiint_{\text{all space}} \Psi^* A_k\Psi\,d\mathcal{V}$$

$$+q\mathbf{E}\iiint_{\text{all space}} \Psi^*\Psi\,d\mathcal{V}$$

$$= -\frac{q}{m}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\varepsilon_{jkl}\boldsymbol{\delta}_j B_k\iiint_{\text{all space}} \Psi^*\left(-i\hbar\frac{\partial}{\partial x_l}\right)\Psi\,d\mathcal{V}$$

$$-\frac{q^2}{m}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\varepsilon_{jkl}\boldsymbol{\delta}_j B_l\iiint_{\text{all space}} \Psi^* A_k\Psi\,d\mathcal{V}$$

$$+q\mathbf{E}(1)$$

Continue the simplification.

$$\begin{aligned}
m\frac{d\langle \mathbf{v}\rangle}{dt} &= -\frac{q}{m}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\varepsilon_{jkl}\boldsymbol{\delta}_j B_k\langle\Psi\,|\,\hat{p}_l\,|\,\Psi\rangle \\
&\quad - \frac{q^2}{m}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\varepsilon_{jkl}\boldsymbol{\delta}_j B_l\langle\Psi\,|\,A_k\,|\,\Psi\rangle \\
&\quad + q\mathbf{E} \\
&= -\frac{q}{m}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\varepsilon_{jkl}\boldsymbol{\delta}_j B_k\langle p_l\rangle \\
&\quad - \frac{q^2}{m}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\varepsilon_{jkl}\boldsymbol{\delta}_j B_l\langle A_k\rangle \\
&\quad + q\mathbf{E} \\
&= \frac{q}{m}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\varepsilon_{jlk}\boldsymbol{\delta}_j \langle p_l\rangle B_k \\
&\quad - \frac{q}{m}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\varepsilon_{jkl}\boldsymbol{\delta}_j (q\langle A_k\rangle)B_l \\
&\quad + q\mathbf{E} \\
&= q\mathbf{E} + \frac{q}{m}(\langle\mathbf{p}\rangle\times\mathbf{B}) - \frac{q}{m}[(q\langle\mathbf{A}\rangle)\times\mathbf{B}] \\
&= q\mathbf{E} + \frac{q}{m}(\langle\mathbf{p}\rangle - q\langle\mathbf{A}\rangle)\times\mathbf{B} \\
&= q\mathbf{E} + q\left(\frac{1}{m}\langle\mathbf{p}-q\mathbf{A}\rangle\right)\times\mathbf{B} \\
&= q\mathbf{E} + q\langle\mathbf{v}\rangle\times\mathbf{B}
\end{aligned}$$

Therefore,

$$m\frac{d\langle \mathbf{v}\rangle}{dt} = q(\mathbf{E} + \langle\mathbf{v}\rangle\times\mathbf{B}).$$